PURELY INFINITE, SIMPLE C^*-ALGEBRAS ARISING FROM FREE PRODUCT CONSTRUCTIONS, III

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Abstract. In the reduced free product of C*–algebras, $(A,\phi)=(A_1,\phi_1)*(A_2,\phi_2)$ with respect to faithful states ϕ_1 and ϕ_2, A is purely infinite and simple if A_1 is a reduced crossed product $B\rtimes_{\alpha,r}G$ for G an infinite group, if ϕ_1 is well behaved with respect to this crossed product decomposition, if $A_2\neq\mathbf{C}$ and if ϕ is not a trace.

The reduced free product construction for C*–algebras was invented independently by Voiculescu [**11**] and, in a more limited sense, Avitzour [**1**]. (The term "reduced" is to distinguish this construction from the universal or "full" free product of C*–algebras.) It is a natural construction in Voiculescu's free probability theory, (see [**12**]). Given unital C*–algebras A_ι with states ϕ_ι whose GNS representations are faithful, $(\iota\in I)$, the construction yields

$$(A,\phi)=\mathop{*}_{\iota\in I}(A_\iota,\phi_\iota),$$

where A is a unital C*–algebra containing copies $A_\iota\hookrightarrow A$ and generated by $\bigcup_{\iota\in I}A_\iota$, and where ϕ is a state on A with faithful GNS representation that restricts to give ϕ_ι on A_ι for every $\iota\in I$ and such that $(A_\iota)_{\iota\in I}$ is free with respect to ϕ. Moreover, ϕ is a trace if and only if every ϕ_ι is a trace; by [**4**], ϕ is faithful on A if and only if ϕ_ι is faithful on A_ι for every $\iota\in I$.

It is a very interesting open question whether every simple, unital C*–algebra must either have a trace or be purely infinite. Purely infinite C*–algebras were defined by J. Cuntz [**3**]. A simple unital C*–algebra A is purely infinite if and only if for every positive element $x\in A$ there is $y\in A$ with $y^*xy=1$. An equivalent condition is that every hereditary C*–subalgebra of A contains an infinite projection.

Let

$$(A,\phi)=(A_1,\phi_1)*(A_2,\phi_2)$$

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be a reduced free product of C*–algebras. In [**8**] it was shown that if ϕ_1 or ϕ_2 is nontracial and if A_1 and A_2 are not too small in a specific sense, then A is properly infinite. It is a plausible conjecture that whenever A is simple and at least one of ϕ_1 and ϕ_2 is not a trace, the C*–algebra A must be purely infinite. The first results in this direction were [**7**], where in a certain class of examples when ϕ_1 was assumed to be non–faithful, A was shown to be purely infinite and simple. In [**5**], assuming ϕ_1 and ϕ_2 faithful, A was shown to be purely infinite and simple in the case when the centralizer of ϕ_1 in A_1 contains a diffuse abelian subalgebra and when A_2 contains a partial isometry that, loosely speaking, scales ϕ_2 by a constant $\lambda \neq 1$. In [**9**], reduced free products of (countably) infinitely many C*–algebras that are not too small in a specific sense were shown to be purely infinite.

In this note, we prove a theorem implying that A is purely infinite and simple under somewhat different conditions. For example, if $A_1 = C(\mathbf{T})$ is the algebra of all continuous function on the circle and if ϕ_1 is given by integration with respect to Haar measure, then A is simple and purely infinite provided only that $A_2 \neq \mathbf{C}$ and ϕ_2 is faithful but not a trace.

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Notation. We begin with some notation, which has appeared elsewhere. Given an algebra $\mathfrak{A}$ and subsets $S_\iota \subseteq \mathfrak{A}$, $(\iota \in I)$ let $\Lambda^o((S_\iota)_{\iota\in I})$ be the set of all words $w = a_1 a_2 \cdots a_n$ where $n \geq 1$, $a_j \in S_{\iota_j}$ and $\iota_1 \neq \iota_2$, $\iota_2 \neq \iota_3$, $\ldots$, $\iota_{n-1} \neq \iota_n$. We will refer to the elements $a_1, \ldots , a_n$ as the letters of the word w; we will sometimes regard the word as a product of specific letters, and sometimes as an actual element of the algebra $\mathfrak{A}$, as it suits the situation.

Moreover, if a C*–algebra A and a state $\phi : A \to \mathbf{C}$ are specified, we will denote by A^o the kernel of ϕ.

Theorem. *Let A_1 be a reduced crossed product C*–algebra, $A_1 = B \rtimes_{\alpha,r} G$, where G is an infinite discrete group and where B is a unital C*–algebra. Denote by u_g, $(g \in G)$ the unitaries in A_1 arising from the reduced crossed product construction and implementing the automorphisms α_g on B. Let ϕ_1 be a faithful state on B that is preserved by all the automorphisms α_g and denote also by ϕ_1 its extension to the state on A_1 that vanishes on the subspace Bu_g for*

every nontrivial $g \in G$. Let A_2 be a unital C–algebra, $A_2 \neq \mathbf{C}$, with a faithful state ϕ_2; let*

$$(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$$

be the reduced free product of C–algebras. Suppose that at least one of ϕ_1 and ϕ_2 is not a trace.*

Then A is purely infinite and simple.

Proof. Our strategy will be to show that A is itself the reduced crossed product of a C*–subalgebra D by the group G, where D is (isomorphic to) the reduced free product of infinitely many C*–algebras; a result from [**9**] will thereby show that D is purely infinite and simple. We will then show that the action of G on D is properly outer; a result of Kishimoto and Kumjian [**10**] will thereby imply that A is purely infinite and simple.

Claim 1. *The family*

$$\big(B, (u_g^* A_2 u_g)_{g \in G}\big)$$

is free with respect to ϕ.

Proof. We must show that

$$\Lambda^{\mathrm{o}}\big(B^{\mathrm{o}}, (u_g^* A_2^{\mathrm{o}} u_g)_{g \in G}\big) \subseteq \ker \phi. \tag{1}$$

Let x be a word belonging to the left–hand–side of (1). Splitting off the unitaries u_g^* and u_g from the letters in x, then grouping together any neighbors in the resulting word belonging to A_1 and using that $u_{g_1} B^{\mathrm{o}} u_{g_2}^* \subseteq A_1^{\mathrm{o}}$ whenever $g_1, g_2 \in G$ and that $u_{g_1} u_{g_2}^* \in A_1^{\mathrm{o}}$ if $g_1 \neq g_2$, we see that x is equal to a word, $x' \in \Lambda^{\mathrm{o}}(A_1^{\mathrm{o}}, A_2^{\mathrm{o}})$. Hence $x \in \ker \phi$ by freeness. This finishes the proof of Claim 1.

Let D be the C*–subalgebra of A generated by $B \cup \bigcup_{g \in G} u_g^* A_2 u_g$.

Claim 2. *D is simple and purely infinite.*

Proof. Since $A_2 \neq \mathbf{C}$ there is a self–adjoint element, $x \in A_2 \backslash \mathbf{C}1$. Let μ be the distribution of x; namely, μ is the probability measure whose support is the spectrum of x and such that $\phi_2(x^k) = \int_{\mathbf{R}} t^k \, d\mu(t)$ for all $k \geq 1$. A consequence of Bercovici and Voiculescu's result [**2**, Prop. 8] is that for some n large enough, the measure arising as the n–fold additive free convolution

$$\mu_n \overset{\text{def}}{=} \underbrace{\mu \boxplus \mu \boxplus \cdots \boxplus \mu}_{n \text{ times}}$$

has support equal to an interval $[a,b]$ and is absolutely continuous with respect to Lebesgue measure. If $g_1, g_2, \dots, g_n$ are distinct elements of G, then by Claim 1 the distribution of $y \overset{\text{def}}{=} \sum_{j=1}^n u_{g_j}^* x u_{g_j}$ is μ_n; therefore y generates an abelian subalgebra of

$$D(g_1, \dots, g_n) \overset{\text{def}}{=} C^*\Big(\bigcup_{j=1}^n u_{g_j}^* A_2 u_{g_j}\Big)$$

on which ϕ is given by a measure without atoms; it follows from [**6**, Prop. 4.1] that $D(g_1, \dots, g_n)$ contains a unitary v satisfying $\phi(v) = 0$; (in fact, this proposition gives $\phi(v^k) = 0$ for all nonzero integers k, but we will not need this). Therefore, partitioning the family $(u_g^* A_2 u_g)_{g \in G}$ into subcollections of cardinality n, and including B in one of these subcollections, we see that D is isomorphic to the free product of infinitely many C*–algebras with respect to faithful states,

$$(D, \phi) \cong \mathop{*}_{k=1}^{\infty} (D_k, \psi_k),$$

where each D_k contains a unitary that evaluates to zero under ψ_k. Moreover, since either ϕ_2 or $\phi_1|_B$ is not a trace, at least one of the ψ_k is not a trace. By [**9**, Thm. 2.1], D is therefore simple and purely infinite. This finishes the proof of Claim 2.

Claim 3. *D has trivial relative commutant in A.*

Proof. Let

$$D_0 = C^*\Big(\bigcup_{g \in G} u_g^* A_2 u_g\Big) \subseteq D;$$

we will show that D_0 has trivial relative commutant in A, which will imply the same for D. Suppose that $x \in A$ and x commutes with D_0; our goal is to show that x must belong to $\mathbf{C}1$. Let $x_0 = x - \phi(x)1$ and suppose, to obtain a contradiction, that $x_0 \neq 0$. Since ϕ is faithful, $\|x_0\|_2 = \phi(x_0^* x_0)^{1/2} > 0$. Choose ϵ so that $0 < \epsilon < \frac{\|x_0\|_2}{3}$. Since

$$\mathbf{C}1 + \operatorname{span} \Lambda^{\text{o}}\Big(B^{\text{o}} \cup \bigcup_{g \in G\setminus\{e\}} B u_g,\ A_2^{\text{o}}\Big) \tag{2}$$

is a dense $*$–subalgebra of A, and since $\Lambda^{\text{o}}(B^{\text{o}} \cup \bigcup_{g \in G\setminus\{e\}} B u_g, A_2^{\text{o}}) \subseteq \ker \phi$, there is a sum of finitely many words, $y = w_1 + w_2 + \cdots + w_m$ with $w_1, w_2, \dots, w_m \in \Lambda^{\text{o}}(B^{\text{o}} \cup \bigcup_{g \in G\setminus\{e\}} B u_g, A_2^{\text{o}})$, such that $\|x_0 - y\| < \epsilon$. Let F be the finite subset of G whose elements are the identity element and all nontrivial elements $g \in G$ for which some w_j has a letter coming from $B u_g$. From the proof of Claim 2, there is $n \in \mathbf{N}$ such that for any n distinct elements, $g_1, g_2, \dots, g_n$ of G, there is a unitary

$$v \in D(g_1, g_2, \dots, g_n) = C^*\Big(\bigcup_{j=1}^n u_{g_j}^* A_2 u_{g_j}\Big)$$

with $\phi(v) = 0$. We take this unitary v having ensured that the n distinct elements satisfy $g_j \notin F$ and $g_j^{-1} \notin F$ for every $j \in \{1, \ldots, n\}$.

Let us show that vy and yv are orthogonal with respect to the inner product on A induced by ϕ, i.e. that $\langle yv, vy \rangle_\phi = \phi(v^* y^* vy) = 0$. Since

$$\mathbf{C}1 + \operatorname{span} \Lambda^{\mathrm{o}}(u_{g_1}^* A_2^{\mathrm{o}} u_{g_1}, u_{g_2}^* A_2^{\mathrm{o}} u_{g_2}, \ldots, u_{g_n}^* A_2^{\mathrm{o}} u_{g_n})$$

is a dense $*$–subalgebra of $D(g_1, \ldots, g_n)$ and since (as can be seen using Claim 1)

$$\Lambda^{\mathrm{o}}(u_{g_1}^* A_2^{\mathrm{o}} u_{g_1}, u_{g_2}^* A_2^{\mathrm{o}} u_{g_2}, \ldots, u_{g_n}^* A_2^{\mathrm{o}} u_{g_n}) \subseteq \ker \phi,$$

for every $\eta > 0$ there is a sum of finitely many words, $z = w_1' + w_2' + \cdots + w_p'$ with

$$w_1', \ldots, w_n' \in \Lambda^{\mathrm{o}}(u_{g_1}^* A_2^{\mathrm{o}} u_{g_1}, u_{g_2}^* A_2^{\mathrm{o}} u_{g_2}, \ldots, u_{g_n}^* A_2^{\mathrm{o}} u_{g_n}),$$

such that $\|v - z\| < \eta$. But we see that each w_j' is equal to a word

$$w_j'' \in \Lambda^{\mathrm{o}}\big(\{u_g \mid g \in G \backslash \{e\}\}, A_2^{\mathrm{o}}\big)$$

where w_j'' begins with $u_{g_j^{-1}}$ and ends with u_{g_k} some $j, k \in \{1, \ldots, n\}$, and where w_j'' has length at least three. Since

$$w_1, \ldots, w_m \in \Lambda^{\mathrm{o}}\Big(B^{\mathrm{o}} \cup \bigcup_{g \in F \backslash \{e\}} B u_g,\ A_2^{\mathrm{o}}\Big),$$

when we consider a product $(w_{i_1}'')^* w_{j_1}^* w_{i_2}'' w_{j_2}$ for arbitrary $i_1, i_2 \in \{1, \ldots, p\}$ and $j_1, j_2 \in \{1, \ldots, m\}$, the choice of the elements $g_1, \ldots, g_n$ ensures that there is not too much cancellation and we are left with a reduced word

$$(w_{i_1}'')^* w_{j_1}^* w_{i_2}'' w_{j_2} = w \in \Lambda^{\mathrm{o}}\Big(B^{\mathrm{o}} \cup \bigcup_{g \in G \backslash \{e\}} B u_g, A_2^{\mathrm{o}}\Big);$$

hence $\phi((w_{i_1}'')^* w_{j_1}^* w_{i_2}'' w_{j_2}) = 0$. This implies that $\phi(z^* y^* z y) = 0$. Since $\eta > 0$ was arbitrary and $|\phi(v^* y^* v y) - \phi(z^* y^* z y)| \leq \eta(2 + \eta)\|y\|^2$, we have $\phi(v^* y^* v y) = 0$, i.e. yv and vy are orthogonal.

We now obtain the contradiction. Since x_0 belongs to the commutant of D_0, we must have $vx_0 - x_0 v = 0$. But by orthogonality of vy and yv,

$$\|vy - yv\| \geq \|vy - yv\|_2 > \|vy\|_2 = \|y\|_2$$

and hence

$$\|vx_0 - x_0 v\| \geq \|vy - yv\| - 2\epsilon > \|y\|_2 - 2\epsilon \geq \|x_0\|_2 - 3\epsilon > 0,$$

which is a contradiction. This finishes the proof of Claim 3.

Claim 4. *For every nontrivial $g \in G$, $\beta_g \overset{\text{def}}{=} \mathrm{Ad}(u_g)$ is an outer automorphism of D, $g \mapsto \beta_g$ is a group homomorphism and A is isomorphic to the reduced crossed product $D \rtimes_{\beta,r} G$.*

Proof. Clearly, $\mathrm{Ad}(u_g)$ is an automorphism of D, for every $g \in G$ and $g \mapsto \beta_g$ is a group homomorphism. From the density of (2) in A and the fact that $u_g B = B u_g$, we see that span $\bigcup_{g\in G} D u_g$ is dense in A. Moreover, whenever $g' \in G$ is nontrivial, $D u_{g'} \subseteq \ker \phi$; this can be seen by approximating an arbitrary element of $D u_{g'}$ by sums of words each belonging to $\{u_{g'}\} \cup \Lambda^{\text{o}}(B^{\text{o}}, (u_g^* A_2^{\text{o}} u_g)_{g\in G}) u_{g'}$. As the GNS representation of ϕ is faithful on A, one sees that A is isomorphic to the reduced crossed product $D \rtimes_{\beta,r} G$.

We will now show that β_g is an outer automorphism of D whenever $g \neq e$. Indeed, if it were inner then letting $v_g \in D$ be such that $\beta_g = \mathrm{Ad}(v_g)$, we would have $u_g^* v_g$ commuting with D. By Claim 3, this would imply that u_g is a scalar multiple of v_g, hence belongs to D, which contradict that $D u_g \subseteq \ker \phi$. This finishes the proof of Claim 4.

Now that A is seen to be the crossed product of a simple, purely infinite C*–algebra by an infinite discrete group acting by outer automorphisms, Kishimoto and Kumjian's result [**10**, Lemma 10] shows that A is simple and purely infinite.

□

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